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Non Zero-Divisors in the Cohomology Ring of a Finite Group

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0. Introduction

In my lecture we discuss Carlson's problem on the index of a module over a finite group algebra. The concept of the index of a module was introduced by Carlson in his lecture at Conference of Tsukuba ICRA '90. There he defined the index in terms of homogeneous systems of parameres for cohomology rings and modules involving exactness conditions on some Koszul complexes. Then he and Benson [5],[6] showed that the index can be defined using the notion of quasi-regular sequences incohomology rings.

It is very interesting problem to calculate the index of the trivial module because it provides many informations on generators and their relations in the cohomology ring of a finite group.

We first prove "Old"theorem on the existence of non zero-divisors in the cohomology ring applying a "new" argument by Benson-Carlson [4]. As a cosequence we can show that the index of the trivial module of a finite simple group of 2-rank 3 is 0.

Next we discuss "relative projective covers" of a module introduced by Knörr [8]. I belive that "relative projective covers" provides a useful tool to investigate the cohomology ring. Asai [2] and Asai-Sasaki [3] used it to calculate the cohomology ring of finite groups with dihedral 2-Sylow subgroups.

1. Index of a Module

Let G be a finite group and k a field of characteristic p 0. For a kG-module M, let $H^n(G,M)=\mathrm{Ext}^n_{kG}(k,M)$ and $H^*(G,M)=\oplus\sum_{n\geq 0}H^n(G,M)$. It is known that $H^*(G,k)$ is a "commutative" noetherian graded ring and $H^*(G,M)$ is a finitely generated graded $H^*(G,k)$-module.

Definition (1.1). Let M be a kG-module and $\rho_1,\ldots,\rho_r$ be homogeneous elements in $H^*(G,k)$ with $\deg\rho_i=r_i$. Then $\rho_1,\ldots,\rho_r$ is said to be a quasi-regular sequence of index t for M if for each $s(1\leq s\leq r)$, the map given as multiplication by ρ_s

$$\tilde{\rho}_s : H^n(G,M)/\Sigma_{i=1}^{s-1}H^{n-r_i}(G,M)\rho_i \longrightarrow H^{n+r_s}(G,M)/\Sigma_{i=1}^{s-1}H^{n+r_s-r_i}(G,M)\rho_i$$

is injective for all $n\geq\Sigma_{i=1}^{s-1}r_i+t$.

A quasi-regular sequence is complete if ,in addition, $\tilde{\rho}_r$ is an isomorphism.

Definition (1.2). [6]. Index of M= min.{t: there exists a complete quasi-regular sequence of index t for M}.

Remark (1.3). At Tsukuba Conference "90 Carlson defined the index by using "Koszul complex" of certain types. The equivalence of these two definitions was proved in Benson=Carlson [5] and Carlson [6].

Concerning the index of a module Carlson asked the following.

Carlson's Problem (1.4). Is the index of k zero ?

As is remarked in [6] the following is known :

Example (1.5). The index of k is zero in the following cases.

(1). The Sylow p-subgroup of G is abelian

(2). $p=2$ and the Sylow 2-subgroup is extra special [9]

(3). $G=GL(n,q^m)$, q a prime $\neq p$ [10]

(4). G has p-rank 2

(5). $p=2$ and G is M_{12}, $G_2(q)$ or ${}^3D_4(q)$, q an odd prime power[1].

It is known [6] that the index of an indecomposable module is at most the index of its source module. In the rest of our discussion we assume that G is a p-group.

2. Non Zero-Divisors

Let ρ be in $H^r(G,k)$. Using an isomorphism $H^r(G,k)\cong \mathrm{Hom}_{kG}(\Omega^r(k),k)$ we regard ρ in $\mathrm{Hom}_{kG}(\Omega^r(k),k)$. Then we obtain an exact sequence:

(2.1) $\quad 0\to L_\rho\to\Omega^r(k)\xrightarrow{\rho} k\to 0$ where $L_\rho=\mathrm{Ker}\rho$.

Tensoring $\Omega^n(k)$ with (2.1) we have an exact sequence $0\to L_\rho\otimes\Omega^n(k)\to\Omega^r(k)\otimes\Omega^n(k)\to\Omega^n(k)\to 0$. Use the fact that $L_\rho\otimes\Omega^n(k)\cong\Omega^n(L_\rho)\oplus$projectives and $\Omega^r(k)\otimes\Omega^n(k)\cong\Omega^{n+r}(k)\oplus$projectives to obtain

(2.2) $\quad 0\to\Omega^n(L_\rho)\to\Omega^{n+r}(k)\oplus P\to\Omega^n(k)\to 0$ where P is projective (or 0). The following is a very useful result ;

Theorem (2.3).[4]. The map given as multiplication by ρ: $H^n(G,k)\to H^{n+r}(G,k)$ is injective if and only if $P=0$.

Using the above theorem we shall prove

Old theorem(2.4). Let Z be a central subgroup of G of order p. If ρ_Z is not nilpotent, then ρ is not a divisor of zero.

Proof. Let $0\to\Omega^n(k)\to P_n\to \dots\to P_1\to k\to 0$ and $0\to\Omega^n_Z(k)\to Q_n \to\dots\to Q_1\to k\to 0$ be projective resolutions of k as a kG-module and as a kG/Z-module respectively. Using the projectivity of P_i's we obtain the following commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n(k) \arrow[r] \arrow[d, "\gamma_n"] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^n_Z(k) \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & k \arrow[r] & 0
\end{tikzcd} .$$

Tensoring $\Omega^n(k)$ and $\Omega^n_Z(k)$ with (2.1) we have a commutative diagram :

$$(2.5)\quad \begin{tikzcd}
0 \arrow[r] & L_\rho\otimes\Omega^n(k) \arrow[r] \arrow[d, "I_{L_\rho}\otimes\gamma_n"] & \Omega^r(k)\otimes\Omega^n(k) \arrow[r] \arrow[d, "I_{\Omega^r(k)}\otimes\gamma_n"] & \Omega^n(k) \arrow[r] \arrow[d, "\gamma_n"] & 0 \\
0 \arrow[r] & L_\rho\otimes\Omega^n_Z(k) \arrow[r] & \Omega^r(k)\otimes\Omega^n_Z(k) \arrow[r] & \Omega^n_Z(k) \arrow[r] & 0
\end{tikzcd} .$$

As ρ_Z is not nilpotent, $L_{\rho}{}_Z$ is projective. Thus $L_\rho\otimes Q_i$ is for all i and therefore $I_{L_\rho}\otimes\gamma_n$ is an isomorphism "modulo projectives So we have from (2.5) the following commutative diagram :

$$(2.6)\quad \begin{tikzcd}
0 \arrow[r] & \Omega^n(L_\rho) \arrow[r] \arrow[d, "\text{iso.}"] & \Omega^{n+r}(k)\oplus P \arrow[r] \arrow[d] & \Omega^n(k) \arrow[r] \arrow[d, "\gamma_n"] & 0 \\
0 \arrow[r] & \Omega^n(L_\rho) \arrow[r] & \Omega^r(\Omega^n_Z(k))\oplus Q \arrow[r] & \Omega^n_Z(k) \arrow[r] & 0
\end{tikzcd}$$

, where P and Q are projective (or zero).

Using the fact that $\Omega^n(L_\rho)_Z$ is projective and $\mathrm{Ker}\Omega^n_Z(k)\supset Z$ we can conclude that Q is 0. By (2.6) we have an exact sequence :

$0\to\Omega^{n+r}(k)\oplus P\to\Omega^r(\Omega^n_Z(k))\oplus\Omega^n(k)\to\Omega^n_Z(k)\to 0$ and it follows that P is 0. Thus by Theorem (2.3) ρ is not a divisor of zero.

We can generalize the above argument to prove the following :

Theorem (2.7). Let r be a p-rank of Z(G). Then there exists a quasi-regular sequence $\rho_1, \dots ,\rho_r$ of index 0 for k such that $L_{\rho_1}\otimes\cdots\otimes L_{\rho_r}\ Z(G)$ is projective.

Using Corollary 9.11[5] we have

Corollary (2.8). If p-rank of $Z(G) \geqq$ p-rank of G -1, then the index of k is zero.

Example (2.9). If $G=U_3(2^m)$ or $Suz(2^m)$, then the index of k is zero for p=2. Using the classification of finite simple groups of 2-rank 3 [7] and the result (1.5) (4),(5) we know that that the index of k is zero for finite simple groups 2-rank $\leqq 3$ for p=2.

3. Relative Projective Covers

$H^r(G,k)=\mathrm{Ext}^r_{kG}(k,k)$. Let ρ be in $H^r(G,k)$ and

(3.1) $0 \to k \to X_r \to \cdots \to X_1 \to k \to 0$ be a corresponding exact sequence of length r.

The following is easy to prove :

Proposition (3.2). For each $n \geqq 0$,

(3.3) $\dim_k H^{n+r}(G,k) \leqq \sum_{i=1}^{r} \dim_k H^{n+i}(G,X_i) + \dim_k H^n(G,k)$.

If "=" holds in (3.3) for all $n \geqq 0$, then ρ is not a divisor of zero.

Proposition (3.4). Assume that in (3.1) for each i, X_i is $\mathcal{X}_i$-projective for some family $\mathcal{X}_i$ of subgroups of G. If "=" in (3.3) holds for all H in $\mathcal{X}_i$ $(1 \leqq i \leqq r)$, then "=" holds for G.

Proof. This follows from Higman's Criterior for $\mathcal{X}_i$-projective modules.

We "want to" have ρ such that

(3.5) For each i , $X_i = \oplus\sum_j k_{H_{ij}}^G$, where $H_{ij} \subsetneq G$ (so that $H^n(G,X_i) \cong \oplus\sum_j H^n(H_{ij},k)$) and

(3.6) "=" in (3.3) holds.

If we have such a ρ , then (I belive) we can apply induction argument to solve Carlson's Problem (1.4). In order to find such a ρ I think that "relative projective covers" of modules will be useful.

Definition (3.7) [8]. Let $\mathfrak{X}$ be a family of subgroups of G and M be a kG-module. Then there exists a unique short exact sequence : $0 \to N \to X \to M \to 0$ such that

(1) X is $\mathfrak{X}$-projective

(2) the sequence is $\mathfrak{X}$-split

(3) N has no $\mathfrak{X}$-projective direct summand.

The sequence is called an $\mathfrak{X}$-projective cover of M and we denote $\Omega_{\mathfrak{X}}(M)$ for N. If $\mathfrak{X}=\{1\}$, then an $\mathfrak{X}$-projective cover is a usual projective cover of a module.

Example (3.8).

(1) $G=\langle x,y: x^2=y^2=(xy)^{2^n}=1\rangle$, a dihedral 2-group of order 2^{n+1}. $\mathfrak{X}=\{\langle x\rangle,\langle y\rangle\}$. Then $\Omega\circ\Omega_{\mathfrak{X}}(k)=k$. We have the following :

$$0 \to k \to kG \to k_{\langle x\rangle}^G \oplus k_{\langle y\rangle}^G \to k \to 0 \qquad \text{(see[2])}$$

(2) G a (generalized) quaternion 2-group. Then $\Omega^4(k)=k$ and we have the following :

$$0 \to k \to kG \to kG\oplus kG \to kG\oplus kG \to kG \to k \to 0 \quad .$$

(3) p=odd, $G=M_m(p)=\langle x,y: x^p=y^{p^{m-1}}=1,\ x^{-1}yx=y^{1+p^{m-2}}\rangle$ $(m\geq 3)$. $\mathfrak{X}=\{\langle x\rangle\}$. Then $\Omega^{2(p-1)}\circ\Omega_{\mathfrak{X}}^2(k)=k$ and we have the following :

$$0 \to k \to kG^{(s)} \underbrace{\to \cdots \to}_{2(p-1)} kG^{(r)} \to k_{\langle x\rangle}^{G} \oplus KG \to k_{\langle x\rangle}^{G} \to k \to 0 \ .$$

(4) $p=\text{odd}$, $G=M(p)=\langle x,y:\ x^p=y^p=[x,y]^p=1=[[x,y],x]=[[x,y],y]\rangle$. $\mathfrak{X}=\{\langle x\rangle,\langle x^i y\rangle,\ i=0,1,\ldots,p-1\}$. Then $\Omega^{2(p-1)}\circ\Omega^2_{\mathfrak{X}}(k)=k$ and we have the following :

$$0 \to k \to kG^{(s)} \underbrace{\to \cdots \to}_{2(p-1)} kG^{(r)} \to \oplus\sum_{A\in\mathfrak{X}} k_A^{G} \oplus kG^{(t)} \to \oplus\sum_{A\in\mathfrak{X}} k_A^{G} \to k \to 0 \ .$$

(5) $p=2$, $G=$ a 2-sylow subgroup of $A_8=SL(4,2)$.

$$G=\left\{\begin{pmatrix}1&a&x&z\\&1&b&y\\&&1&c\\&&&1\end{pmatrix}\right\},\ H_1=\left\{\begin{pmatrix}1&0&x&0\\&1&b&y\\&&1&0\\&&&1\end{pmatrix}\right\},\ H_{21}=\left\{\begin{pmatrix}1&a&x&0\\&1&b&0\\&&1&0\\&&&1\end{pmatrix}\right\},\ H_{22}=\left\{\begin{pmatrix}1&a&x&0\\&1&x+y&y\\&&1&a\\&&&1\end{pmatrix}\right\},$$

$$H_{23}=\left\{\begin{pmatrix}1&0&0&0\\&1&b&y\\&&1&c\\&&&1\end{pmatrix}\right\},\ H_3=\left\{\begin{pmatrix}1&0&0&0\\&1&b&0\\&&1&0\\&&&1\end{pmatrix}\right\},\ H_4=\left\{\begin{pmatrix}1&a&0&0\\&1&0&0\\&&1&c\\&&&1\end{pmatrix}\right\},\ a,b,c,x,y,z\in GF(2).$$

$\mathfrak{X}_1=\{H_1\}$, $\mathfrak{X}_2=\{H_{21},H_{22},H_{23}\}$, $\mathfrak{X}_3=\{H_3\}$, $\mathfrak{X}_4=\{H_4\}$. Then $\Omega_{\mathfrak{X}_4}\circ\Omega_{\mathfrak{X}_3}\circ\Omega_{\mathfrak{X}_2}\circ\Omega_{\mathfrak{X}_1}(k)=k$ and we have the following :

$$0 \to k \to k_{H_4}^{G} \to k_{H_3}^{G} \to k_{H_{21}}^{G} \oplus k_{H_{22}}^{G} \oplus k_{H_{23}}^{G} \to k_{H_1}^{G} \to k \to 0 \quad .$$

All the above examples satisfy coditions (3.5) and (3.6). We use Proposition (3.4) to check the condition (3.6).

Remark (3.9). We can show that for A_8 , $p=2$, the index of k is zero by Example (3.8),(5). The cohomology ring of A_8 and its 2-Sylow subgroup are already calculated by Tezuka-Yagita [11].

4. Extra Special p-Groups

We conclude our discussion with the following observation on which I could not talk in my lecture.

Let G be an extra special p-group of order p^{2r+1}.

If $p=2$, then

(4.1) $G=D_8\underbrace{*\cdots*}_{r}D_8$, the central product of r copies of a dihedral 2-group D_8 of order 8 or

(4.2) $G=Q_8*D_8\underbrace{*\cdots\cdot*}_{r-1}D_8$, the central product of a quaternion 2-group Q_8 of of order 8 and $(r-1)$ copies of D_8.

Put $\mathfrak{X}_s=\{P\subseteq G:\ P\cap Z(G)=1,\ |P|=2^s\}$. Then we can show the following :

Proposition (4.3).

(1) In case (4.1) $\prod_{s=0}^{r-1}\Omega_{\mathfrak{X}_s}^{2^{r-1-s}}\circ\Omega_{\mathfrak{X}_r}(k)=k.$

(2) In case (4.2) $\prod_{s=0}^{r-2}\Omega_{\mathfrak{X}_s}^{2^{r-s}}\circ\Omega_{\mathfrak{X}_{r-1}}^{4}(k)=k.$

Taking relative projective covers of k as in Proposition (4.3) we obtain in case (4.1) (resp. (4.2)) an exact sequence of length $n=2^r$ (resp. 2^{r+1}) of the form :

$0\to k\to X_n\to\cdots\to X_1\to k\to 0$ where X_i is $\mathfrak{X}_{s_i}$-projective for some s_i. This sequence satisfies (3.5).

Question (4.4). Does the above sequence satisfy (3.6) ?

If p is odd, then let $\mathfrak{X}_s=\{P\subseteq G:\ P\cap Z(G)=1,\ |P|=p^s\}$.

Question (4.5).

(1) Is $\prod_{s=0}^{r-1}\Omega_{\mathfrak{X}_s}^{2(p-1)p^{r-1-s}}\circ\Omega_{\mathfrak{X}_r}^{2}(k)=k$?

(2) the same question as in (4.4).

Remark (4.6). If r=1, then questions (4.4) and (4.5) are answered in Example (3.8).

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